Error Analysis of the Finite Element Method for the 1-Dimensional Camley Model

Ayumu Masaki

1st-year Master's student of Natural Science and Technology,
Sakakibara Lab. at Kanazawa University

March 17, 2026

Outline

1. Introduction
2. Mathematical Model and Variational Formulation
3. Error Analysis
4. Numerical Experiments
5. Conclusion

Introduction

Background

- **Cell Polarity:** Concentration differences within the cell drive cell movement.
- **Wave-Pinning Model:** A reaction-diffusion system where the interaction between an active form u and an inactive form v creates a concentration gradient inside the cell.
- **Camley Model:** A model describing cell movement based on the wave-pinning model.

Challenges

- Requires numerical computation on moving domains (e.g., ALE method).
- Rigorous mathematical analysis on the effect of moving meshes on the convergence of solutions is insufficient.

Objective of this Study

To construct a finite element scheme for the 1D Camley model and **establish its error estimates from both theoretical and numerical perspectives**.

Mathematical Model

Domain: $\Omega(t) = (a(t), b(t))$, Outward normal velocity on the boundary: V, $f(u, v)$: Nonlinear term (bistable type), α, β, D_u, D_v: Positive constants

Partial Differential Equations in the domain:

$$\begin{cases} \partial_t u = D_u \partial_x^2 u + f(u, v) & \text{in } \Omega(t), \\ \partial_t v = D_v \partial_x^2 v - f(u, v) & \text{in } \Omega(t). \end{cases} \tag{1}$$

Boundary Conditions (Natural Neumann B.C. on the moving domain):

$$\begin{cases} -D_u \partial_x u + uV = 0 & \text{on } a(t), \\ D_u \partial_x u + uV = 0 & \text{on } b(t), \\ -D_v \partial_x v + vV = 0 & \text{on } a(t), \\ D_v \partial_x v + vV = 0 & \text{on } b(t). \end{cases}$$

Domain Dynamics:

$$\begin{cases} V(a(t), t) = -\dot{a}(t) = \alpha u(a(t), t) - \beta, \\ V(b(t), t) = \dot{b}(t) = \alpha u(b(t), t) - \beta \end{cases}$$

Variational Formulation

Introducing an artificial velocity field w and the material derivative $\partial^{\bullet(w)} u := \partial_t u + w\partial_x u$,
the weak formulation is defined as follows:

Problem: Find $u, v \in L^2(0,T; H^1(\Omega(t)))$ such that for all $t \in [0,T]$

$$\begin{cases} \left(\dfrac{d}{dt}\displaystyle\int_{\Omega(t)} u\xi dx\right) + \displaystyle\int_{\Omega(t)} \left[-u\partial^{\bullet(w)}\xi + uw\partial_x\xi + D_u(\partial_x u)(\partial_x\xi) - f(u,v)\xi\right] dx = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \xi \in H^1(\Omega(t)), \\ \left(\dfrac{d}{dt}\displaystyle\int_{\Omega(t)} v\varphi dx\right) + \displaystyle\int_{\Omega(t)} \left[-v\partial^{\bullet(w)}\varphi + vw\partial_x\varphi + D_v(\partial_x v)(\partial_x\varphi) + f(u,v)\varphi\right] dx = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \varphi \in H^1(\Omega(t)). \end{cases} \tag{2}$$

Error analysis directly on a moving domain is difficult.
$\Rightarrow$ Define a mapping $q(\hat{x}, t)$ to a reference (fixed) domain $\hat{\Omega}$,
and reformulate (2) for the analysis.

Variational Formulation on the Reference Domain

(2) is reformulated on $\hat{\Omega} = (0,1)$ using the mapping $q(\hat{x},t) = a(t)(1-\hat{x}) + b(t)\hat{x}$ as follows:

Find $\hat{u}, \hat{v} \in L^2(0,T;H^1(\hat{\Omega}))$ with $\partial_t \hat{u}, \partial_t \hat{v} \in L^2(0,T;L^2(\hat{\Omega}))$
such that for all $t \in [0,T]$

$$\begin{cases} \displaystyle\int_{\hat{\Omega}} \left[(\partial_t \hat{u})\hat{\xi} l(t) + \hat{u}\hat{\xi}\dot{l}(t) + \hat{u}\hat{w}\partial_{\hat{x}}\hat{\xi} + \frac{D_u}{l(t)}(\partial_{\hat{x}}\hat{u})(\partial_{\hat{x}}\hat{\xi}) - f(\hat{u},\hat{v})\hat{\xi} l(t) \right] d\hat{x} = 0 \\ \qquad\qquad \forall \hat{\xi} \in H^1(\hat{\Omega}) \\ \displaystyle\int_{\hat{\Omega}} \left[(\partial_t \hat{v})\hat{\varphi} l(t) + \hat{v}\hat{\varphi}\dot{l}(t) + \hat{v}\hat{w}\partial_{\hat{x}}\hat{\varphi} + \frac{D_v}{l(t)}(\partial_{\hat{x}}\hat{v})(\partial_{\hat{x}}\hat{\varphi}) + f(\hat{u},\hat{v})\hat{\varphi} l(t) \right] d\hat{x} = 0 \\ \qquad\qquad \forall \hat{\varphi} \in H^1(\hat{\Omega}) \end{cases} \tag{3}$$

Spatial discretization is performed in almost the exact same manner.

Semi-discrete Problem

Discretized as follows using the finite element space $\hat{\Psi}_h := \{\eta \in C(\bar{\hat{\Omega}}) \mid \eta|_{[\hat{x}_i, \hat{x}_{i+1}]} \in \mathbb{P}_1([\hat{x}_i, \hat{x}_{i+1}])\}$ on the reference domain $\hat{\Omega} = (0, 1)$.

$$
\begin{cases}
\displaystyle\int_{\hat{\Omega}} \left[(\partial_t \hat{u}_h)\hat{\xi}_h l_h + \hat{u}_h \hat{\xi}_h (\partial_{\hat{x}} \hat{w}_h) + \hat{u}_h \hat{w}_h (\partial_{\hat{x}} \hat{\xi}_h) + \frac{D_u}{l_h} (\partial_{\hat{x}} \hat{u}_h)(\partial_{\hat{x}} \hat{\xi}_h) - f(\hat{u}_h, \hat{v}_h)\hat{\xi}_h l_h \right] d\hat{x} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \hat{\xi}_h \in \hat{\Psi}_h \\
\displaystyle\int_{\hat{\Omega}} \left[(\partial_t \hat{v}_h)\hat{\varphi}_h l_h + \hat{v}_h \hat{\varphi}_h (\partial_{\hat{x}} \hat{w}_h) + \hat{v}_h \hat{w}_h (\partial_{\hat{x}} \hat{\varphi}_h) \quad + \frac{D_v}{l_h} (\partial_{\hat{x}} \hat{v}_h)(\partial_{\hat{x}} \hat{\varphi}_h) + f(\hat{u}_h, \hat{v}_h)\hat{\varphi}_h l_h \right] d\hat{x} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \hat{\varphi}_h \in \hat{\Psi}_h
\end{cases}
\tag{4}
$$

Here, $\hat{\psi}_i(\hat{x})$ are the basis functions on $\hat{\Omega}$, and

$$
\hat{u}_h(\hat{x}, t) = \sum_i^N u_i(t)\hat{\psi}_i(\hat{x}), \quad \hat{v}_h(\hat{x}, t) = \sum_i^N v_i(t)\hat{\psi}_i(\hat{x}), \quad \hat{w}_h(\hat{x}, t) = \sum_i^N w_i(t)\hat{\psi}_i(\hat{x}),
$$

$$
l_h(\hat{x}, t) = \partial_{\hat{x}} q(\hat{x}, t), \quad \hat{w}(\hat{x}, t) = \partial_t q(\hat{x}, t).
$$

Simulation Results

Simulation Settings:

- **Domain:** $\Omega(0) = (0, 10)$, $t \in [0, 15]$.
- **Initial Condition:** The following function with polarity on the left side is adopted for u:

$$u(x,0) = \begin{cases} 1.5 & (x \leq 1) \\ 0.4 & (x > 1), \end{cases} \qquad v(x,0) = 1.0,$$

- **Number of elements:** $N = 51 \quad (h \approx L(0)/N)$
- **Parameters:** $\alpha = 0.5, \quad \beta = 0.378, \quad D_u = 0.1, \quad D_v = 10.0$

Results:

- The domain $l(t)$ moves to the left over time.
- The domain length $l(t) = 10$ is kept constant by adjusting parameters α and β.
- The boundaries move according to the polarity of u (blue), reaching a stable shape.
- The domain continues to move maintaining a stable shape after $t = 15$.

Error Estimates

By applying the energy method to (3) and (4),
we obtained the following main theorem.

Theorem (Error Estimates)

Assume $u, v \in L^2(0,T;H^2(\Omega))$.
Then, for a sufficiently small mesh size h, the following holds:

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \leq Ch \tag{5}$$

$$\|v - v_h\|_{L^2(0,T;H^1(\Omega))} \leq Ch \tag{6}$$

where C is a positive constant independent of h.

First-order convergence ($O(h)$) is theoretically guaranteed in the $L^2(0,T;H^1(\Omega))$ norm.

Numerical Experiment Settings

Experiments were conducted to verify the validity of the proposed method.
In particular, to confirm consistency with the theoretical analysis,
initial values were set considering the regularity of the solution.

- **Domain:** $\Omega(0) = (0, 10)$, $t \in [0, 5]$.
- **Initial Condition:** A smooth function is adopted (to guarantee $u \in H^2(\Omega)$).

$$u(x, 0) = u_{low} + \frac{u_{high} - u_{low}}{2} \left(1 - \tanh \left(\frac{x - x_c}{w} \right) \right)$$

- **Number of elements:** $N = 51, \ldots, 801.$ $\quad (h \approx L(0)/N)$
- **Time step:** Controlled so that $\Delta t = Ch^2$ $\quad (C = 0.025)$.

Experiment Results

Table 1: Errors and Experimental Orders of Convergence

Step	u in H^1		v in H^1		u in L^2		v in L^2	
$(N_{coarse} \to N_{fine})$	Error	Order	Error	Order	Error	Order	Error	Order
$51 \to 101$	2.85e-01	0.98	3.34e-03	1.07	1.81e-02	1.97	1.38e-03	1.85
$101 \to 201$	1.45e-01	0.99	1.59e-03	1.02	4.61e-03	1.99	3.83e-04	1.96
$201 \to 401$	7.28e-02	1.00	7.81e-04	1.00	1.16e-03	2.00	9.83e-05	1.99
$401 \to 801$	3.65e-02	-	3.89e-04	-	2.91e-04	-	2.48e-05	-

$\Rightarrow$ Confirmed **first-order convergence** in the H^1 norm
and **second-order convergence** in the L^2 norm for both u and v.

Conclusion and Future Work

Contributions of this Study

- Conducted an error analysis using the finite element method for the 1D Camley model.
- Theoretically proved $O(h)$ **convergence** in the $L^2(0,T;H^1(\Omega))$ norm.
- Numerical experiments also yielded $O(h)$ convergence accuracy, consistent with the theory.

Future Work

- Rigorous proof of the existence and uniqueness of the solution for the continuous problem.
- Proof of second-order convergence in $L^2(\Omega)$ using the Aubin–Nitsche trick (duality argument).
- Investigation of the 2D Camley problem.

References I


P. Pozzi and B. Stinner, Curve shortening flow coupled to lateral diffusion, *Numer. Math.* **135** (2017), 1171–1205.


K. S. Putri, T. Mizuochi, N. Kolbe, and H. Notsu, Error estimates for first- and second-order Lagrange-Galerkin moving mesh schemes for the one-dimensional convection-diffusion equation, *J. Sci. Comput.* **101** (2024), Paper No. 37.